Do We Really Want To Keep the Traditional Algorithms for Whole Numbers?
Draft – Not to be Reproduced
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The traditional algorithms for whole-number computation have been a large and constant feature of the elementary school mathematics curriculum for at least a century. In the intervening years there have been changes in both the way that students are taught the algorithms and the algorithms themselves. But today, students still carry and borrow, remember to move over one place in the second row of multiplication and labor over how many times 43 goes into 175. We accept all of this almost blindly due to a long-standing tradition. Parents expect their children to be taught these skills, probably not so much because they feel they will need them in their adult jobs but because it is what has always been done in math classes as long as they can remember.

A few researchers, most notably Constance Kamii (Kamii & Dominick, 1998), have argued against teaching the traditional computation methods. Among the NSF "Standards-based" curricula, the *Investigations in Number, Data, and Space* program has resisted pressures to teach the traditional algorithms. Nonetheless, there has been little real debate among mathematics educators to make any change. Therefore, the purpose of this paper is to encourage debate of the following proposition:

The traditional whole-number algorithms for addition, subtraction, multiplication, and division should no longer be taught in school.

In this paper, I will present arguments for the acceptance of the proposition and leave the opposition to others. Negative arguments generally revolve around the perceived need for efficiency and are motivated by a fear that efficient computational skills will not accrue without the traditional methods.

The arguments in favor of the proposition come under four headings: Problems with the traditional algorithms, the lack of need for the traditional methods in today's society, the proven existence of effective alternatives, and the values that can be attained from these alternative strategies.

PROBLEMS: THE TRADITIONAL ALGORITHMS ARE NOT SERVING US WELL.

The four traditional algorithms are each clever procedures for computation that have been devised to work for all numbers with attention only to basic facts. Some of the problems with these methods are inherent in the algorithms themselves. Others are related to the difficulties of teaching these algorithms in light of their reduced need in today's society.

The traditional algorithms are digit-oriented rather than number-oriented.

With the exception of long division, the algorithms require only consideration of individual digits within the computation. The actual numbers involved are not attended to until the result has been obtained. For example, for the sum 28 + 46, the first addition is 8

+ 6. The 14 must be separated into 1 ten ("carried" to the tens column) and 4 ones which are recorded.

The process moves to the tens column: 1 + 2 + 4. The focus is on these digits, not the numbers 10, 20, and 40. After recording this sum, the result of 74 can be observed. In contrast, one alternative strategy would be to add 20 and 40 and then 8 and 6. The resulting two sums, 60 and 14 are then combined, again thinking of them as complete numbers.

A similar contrast is seen in multiplication. An alternative, number-oriented strategy for 68 x 7 begins with 7 x 60 is 420 and then 56 more is 476. The first product is 7 times *sixty,* not the digit 6.

With long division, students are taught to temporarily ignore the left digits of the dividend until they are needed. For example, to compute 538 ÷ 7, students are taught to cover up the digit 8 either mentally or with their finger and think about 53 ÷ 7 as if the 8 were not there.

A number-oriented approach might begin by thinking about what times 7 is close to 538 -- 7 x 70 is 490 and 7 x 80 is 560, too much. The quotient is between 70 and 80.

The traditional algorithms are right-handed instead of left-handed.
The digit-orientation of the traditional algorithms begins with the least-significant digits – those on the right. When subtracting 73 – 38, the first thing we know about the result is that it ends in 5. However, a number-oriented approach might suggest that 30 more than 38 is 68 (close to 73). The first thing that is known here is that the difference is a bit more than 30.

Similarly, for the product 26 x 47 the traditional algorithm begins with 7 x 6 telling us that the answer ends in 2. An alternative approach might begin with 20 x 40 or 25 x 40. These first steps tell us about the magnitude of the product.

Currently, when we teach the important skill of estimation, we talk about "front-end" strategies. That is, we have to label and overtly attend to the idea of beginning with the most important parts of the numbers in contrast to the algorithms students have been taught. Virtually every alternative strategy for computation is "left-handed" or begins with the whole number. There is no need to unteach an algorithm if we don't teach it to begin with.

The traditional algorithms must be taught in a teacher-directed manner.
It is interesting to look at books and articles about whole-number computation over just the last 30 years. There is little doubt that we have come a long way in helping students understand the traditional algorithms. The curriculum also has been adjusted so that the extremely tedious computations of the 1950s are no longer with us. Having said that, there is no chance that the ingenious traditional methods that have been invented and refined over time are ever going to be "invented" by students. Even with a very

conceptual orientation, we must tell students how to do the recording for each step and in what order. The rules involving carried digits, borrowing across zero, moving the product of the tens digit one space to the left, and placing digits correctly in the dividend do not come easy to students, even with very careful conceptual development. And even the best of teacher intentions go astray after the first few days of teaching with the use of base-ten materials or other models designed to provide understanding. Repetitious drills consume most of students' time with computation. And after the last 50 years or so of improvement in instruction and a host of well-intentioned approaches, we still have a lot of students making systematic errors. More importantly, the amount of time that is spent following the rules of the algorithms as directed by the teacher sends a faulty view of what it means to "do mathematics." Mathematics is the "science of pattern and order" (*Everybody Counts,* MSEB, 1989). Mathematics is about making sense of numbers and patterns in our world. It is not about following rules – even if those rules can be taught conceptually.

Students make more errors with traditional algorithms than with their invented strategies.

Not only do students make errors when they use the traditional algorithms, these errors are generally systematic or "buggy algorithms" that they tend to use again and again. The cause of these faulty algorithms is easily traced to a lack of full understanding. This continues to be true today even though our knowledge of how to teach the traditional algorithms is better now than it was 30 years ago. In contrast, when students utilize alternative strategies that they themselves have invented or that they have acquired from a peer, the methods tend to be better understood. "When students fail to grasp the concepts that underlie procedures or cannot connect the concepts to the procedures, they frequently generate flawed procedures that result in systematic patterns of errors. … When the initial computational procedures that students use to solve multidigit problems reflect their understanding of numbers, understanding and fluency develop together" (*Adding It Up,* National Research Council, 2001, p. 196).

The traditional algorithms are unnecessarily tedious due to lack of flexibility.

With the traditional algorithms, one size fits all. Over the years different algorithms have been introduced, notably for subtraction (equal additions instead of decomposition) and long division (a repeated subtraction or Greenwood method instead of a partitioning approach). Even so, never has the mainstream curriculum encouraged students to use a variety of algorithms for a single operation with the intent that they would adapt strategies to the particular numbers involved. For example, when the difference 2000 – 125 is required, even most adults –conditioned to a single traditional algorithm – will instinctively line up the 125 beneath the zeros and borrow from the 2:
In contrast, if one has become accustomed to a number-oriented approach to computation, it is reasonable to think 2000 minus 100 is 1900 and another 25 is 1875. This can be done mentally or the intermediate result of 1900 can be written down to avoid short-term- memory problems.

As another example, when a divisor is a "nice" number such as 25, the traditional algorithm still has us doing our "guzintas." Consider 487 ÷ 25. After the first division, subtraction, and bring down, we only know that the answer has a 1 in the tens digit and we still are faced with 237 ÷ 25.

2 0 0 0
– 1 2 5
1 9 9
1
25 4 8 7
2 5
2 3 7

In contrast, when dividing by 25, thinking of a missing factor is reasonably easy. Four 25s are 100, eight are 200, 16 are 400, and 19 are 475. There are now 12 remaining in the original 487 so the answer is 19 with 12 left over or just a bit less than 19 1/2. In general, a missing factor approach to division is essentially all that is required for everyday usage.

The traditional algorithms do not lend themselves to mental methods or to estimation.

The mainstream curricula have, in recent years, moved instruction with mental mathematics and estimation to just before work with the traditional algorithms. This is because it is difficult to have students practice using the right-handed, digit-oriented methods of the traditional algorithms and then follow immediately with left-handed, number-oriented approaches used in mental computations. However, this simple shift simply does not work. In second grade, work on mental strategies only just begins and then students are taught the good old pencil-and-paper method of adding and subtracting. The mental strategies do not get practiced.

Beyond the second grade, students are already engrained in the traditional procedures. Regardless of where mental computation falls in the textbook, students must somehow forget the “real” way to compute and attempt the number-oriented, left-handed methods of mental computation to which they have had only sparse exposure at best. A week later, the traditional algorithm is there again to be practiced some more.

In the twenty-first century mental computation is far more valued than it was 30 years ago. It is a major component of number sense. It allows for ease in problem solving and for dealing with many everyday situations when pencil and paper are simply not reasonable.

The following are attributes of mental computation – and by extension, estimation, since estimation is simply a mental computation performed on close but easier-to-use numbers:

Mental computation -

• Is number oriented rather than digit oriented.
• Typically begins with large chunks of the computation and deals with the refinements last. In other words, it is left-handed.
• Never involves regrouping! Instead of regrouping, mental computations work up or down through the next multiple of 10 or 100 as appropriate.

The above list of attributes is precisely the same list that can be applied to number-oriented or “invented” strategies. When students initially develop or adopt these methods, they usually support their thinking with written work. With sufficient use, the need for written support diminishes and for many computations, the methods can be carried out mentally. The current distinction that is made between mental mathematics and invented or alternative algorithms is unfortunate. Rather, the focus should be on flexible, number-oriented methods that can (and for the most part, will) become mental with practice and familiarity.

It makes no instructional sense to teach one set of algorithms for pencil-and-paper procedures and a significantly different and conflicting set of procedures for mental computation and estimation. A focus on the number-oriented, flexible methods can do it all.

We no longer can afford the time required for teaching, reteaching, and remediation of outmoded skills.

Although mathematics education has come a very long way in the effective teaching of the traditional algorithms, the fact remains that a huge portion of the elementary curriculum is spent on the traditional algorithms. Much of that time is devoted to reteaching what is forgotten over the summer, extending the algorithms to yet more digits, and remediation for those students who continue to make errors. Our efforts at connecting conceptual understanding to the traditional algorithms is perhaps praiseworthy yet certainly not adequate. In mainstream curricula, the conceptual development is typically very short – amounting to a lesson or two per operation. The predominant efforts are based on practice of skills. If this same time were spent on number-oriented, flexible computation, not only would students acquire adequate skills, we would not need to teach separate methods for mental computation and estimation.

NEED: THE TRADITIONAL ALGORITHMS ARE NO LONGER NEEDED IN TODAYS SOCIETY

Although nearly every parent seems to want their child to be taught the same arithmetic skills that they and their parents before them were taught, the reality today is that very little pencil-and-paper arithmetic is done outside the confines of the classroom. This is not to say that the average citizen does not need to be able to compute. It is the nature of computational needs that has changed. To examine this claim, it is useful to separate computation in the workplace from that done in everyday situations outside of work.

In the workplace, technology does virtually all of the required computation.

One of the most compelling arguments for schooling in general is to prepare students to be profitably employed outside of school. For most of our nation’s history, all accurate computation had to be done by hand. Logarithms were used to convert multiplication and division to easier addition and subtraction. The slide rule was popular with engineers but even an expensive slide rule could provide only four significant digits and decimal points had to be inserted through estimation. It was only about 35 years ago that calculators became readily available. Today, the four-function calculator is unbelievably inexpensive and does not rely on batteries. Computers are adapted to every repetitive computational situation from the retail check-out, to the accountant’s desk, to the construction trailer, to the science laboratory. As Devlin (1998) put it, "When the automobile became widely available, skill at riding a horse was replaced by skill at driving a car. Likewise, in the age of the pocket calculator and the electronic computer, computational skill is no longer necessary. We need other abilities. … Training students to be a poor imitation of a $30 calculator is a waste of time for both teacher and students." If the job requires accurate computations, any reasonable employer will provide the appropriate calculator or computer in order to assure accuracy and to save valuable time. It would be foolish to do otherwise. Among the "other skills" that mathematician Devlin alludes to are the ability to compute mentally and to estimate. Good workers in almost every field need number sense. Today's curriculum short changes students in these necessary skills while wasting time teaching those skills that are obsolete in the workplace.

Outside of the workplace, number-oriented, flexible strategies are more than adequate.

This may be the most subjective of the arguments. Three things should be kept in mind. First, by advocating that the standard algorithms be dropped, I am not arguing that no calculation be taught. To the contrary, students need significant amount of time, instruction, and practice to develop the flexible computational skills I believe are important. Second, simple four-function calculators are so cheap that they are often given away as sales incentives at fast food restaurants. Every home should have several. Finally, if we are honest, very little difficult computation is every required outside of the workplace. Flexible methods can more than adequately fill nearly every need. Failing that, a calculator is sure to be close by.

Students who have not been taught the traditional algorithms do about as well on standardized tests as do students in traditional programs.

Although it seems a strange argument, standardized testing is often held up as a reason for the focus on computational skills. Should we really be designing our curriculum to serve test scores? I believe testing should be designed to match our curriculum. However, in the late 90s and early into the 21st century, data has been collected from school systems where all of the students have used either *Everyday Mathematics* (before it began to include the traditional algorithms) or *Investigations* and comparisons have been made with comparable schools using traditional programs. In every case, students in these "Standards-based" programs outperform their traditional program counterparts on measures of understanding and problem solving. In the area of multi-digit computation, most studies find that the Standards-based students are roughly on a par with students in traditional programs or outperform them (Fuson, 2003). For other studies, see Campbell, (1996); Carroll, (2000); Mokros, Berle-Carman, Rubin, and O'Neil, (1996); Riordan, & Noyce (2001).

The curriculum advocated by the Freudenthal Institute in the Netherlands has long promoted student-invented computational procedures with much success. Torrence, an American mathematics professor, after her son was exposed to two systems, vividly described the contrast between the Dutch schools' approach to computation and that of the U. S. (Torrence, 2003). As a third grade student, Torrence's son quite readily did multi-digit computations mentally, including multiplication tasks such as 300 x 17.

However, this boy later experienced significant difficulty with simpler computations using the traditional algorithms in his American classroom.

BETTER IDEAS: EFFECTIVE ALTERNATIVES TO THE TRADITIONAL ALGORITHMS ALREADY EXIST

I will repeat that the argument for removing the traditional algorithms from the curriculum is not an argument against computational skills. Nor is it an argument in favor of a dependence on calculators. Computational fluency remains a significant goal for all students, both as a prerequisite for the world of work and as a tool for learning mathematics. As stated in the NCTM *Standards*, “Equally essential [with basic facts] is computational fluency – having and using efficient and accurate methods of computing” (NCTM, 2000, p. 32). Those programs and researchers who have helped students “invent” alternative computational strategies have shown us a lot about what must be done to help students achieve computational fluency without recourse to the traditional algorithms.

Decrease the emphasis on a digit orientation to place-value development.

In 1975, Payne and Rathmell described a structure for place-value development that was designed to help students connect a conceptual understanding of place value with the written and spoken representations of multi-digit numbers. As shown here, this structure illustrated the importance of connecting a conceptual understanding of multidigit numbers with both a written and spoken representation of this concept. The bottom of the triangle connects the oral and written representations. Both then and now, students are helped to develop the concepts of place value with some form of proportional model – base-ten units, strips and squares, bundles of sticks, counters that can be grouped, and so on. Today, all of these models continue to be used. Perhaps the most ingenious being ***Digi Blocks***™.

The Payne-and-Rathmel model remains the predominant guide for place-value development. It was embellished by Thompson (1990) to include a variety of counting techniques (by ones, by groups of tens and singles, and the traditional counting –*ten, twenty, thirty, thirty-one, thirty-two, …*). This is essentially the model that has been found in my textbook for teachers (Van de Walle, 2004) since 1990. So, what is wrong with this approach? Although there is much to commend in the efforts to help students connect representations of base-ten concepts with written and oral forms for numbers, the main flaw in the approach is the failure to help students develop a sense of how any number is connected to close, important “landmark” numbers such as multiples of 10, 25, and 100. In essence this is a static view of numbers. A large portion of place valued development can be summarized in activities such as the following: is ______ tens and ______ ones is ______ in all. 3,746 ---- Circle the digit in the hundreds place. How many tens? Write the expanded form of 643. Trading activities.

The last item above, trading activities, became very popular in the 1970s and 1980s. These activities represent the one aspect of place-value instruction that is not static. Students accumulate or remove base-ten models on/from a place value mat by rolling number cubes or some other method. In the accumulate version, students learn to trade 10 pieces in one position for 1 in the position to the left. The reverse is true for the remove version. Trading activities are the essence of carrying and borrowing in the addition and subtraction algorithms. It would seem that students do focus on how far a number such as 46 is from 50, but that is not the case. The attention is at best on the ones digit. However, even here there is little or no reflection on the fact that 6 is 4 from 10 much less that 46 is 4 from 50 and 54 from 100. In this example, the ones place is only important *after* there are 10 or more pieces there, not before. For example, if 6 ones are added to the 4 tens and 6 ones, student dutifully count 10 of the ones, and trade them for a ten with almost no thought given to the numbers involved. Even the fact that the tens piece is composed of ten ones is of little consequence. Students pay almost no attention to the actual size of these models. So, although trading is not a static approach to place value it is extremely digit-oriented rather than number-oriented. Trading is at best a readiness for carrying and borrowing and probably does very little to help students understand numbers or develop number sense. Furthermore, carrying and borrowing are not ever found in mental computation or the many different strategies found under the heading of flexible, number-oriented approaches. Consider three of many possible methods of adding 38 + 25:

(a) 38 + 20 is 58 and 5 more (2 to get to 60 and 3 more) is 3
(b) 38 and 2 (from the 25) is 40 and then the remaining 23 is 63
(c) 30 and 20 is 50, 8 and 5 is 13, 50 and 13 is 63.

In (a) and (b) the distance from either 58 or 38 is important but no regrouping takes place. In (c), the 13 is never thought of as a single ten and 3 ones but only as a whole number that is easily added to a multiple of ten. Similar analysis will illustrate that there is no regrouping in subtraction methods, especially those that involve adding up rather than take away. But even for take-away strategies, the focus is on the next multiple of ten. For 73 – 38 you might take off 40 and add back 2. Another method would be to first remove 30 to get to 43 and then take off 8: 3 down to 40 and 5 more is 35.
Methods for multiplication and division are all based on strategies used in addition and subtraction. The bottom line is this: if we don't teach students the traditional algorithms we have no need to ever teach regrouping.

Develop "ten-structured" thinking.

To decrease the digit-orientation to place value does not necessarily imply that the triangle structure of concepts-symbols-words should be abandoned. However, it does suggest that we have perhaps over-estimated the value of the very popular units, longs and flats model for numbers. To adults who understand multidigit numbers these models are intuitively attractive as a teaching tool. However, they tend to accentuate a digit-orientation at the expense of the full number being modeled and more importantly the relationship of that number to other important landmark numbers. In place of the time spent on trading and most if not all of the show and tell activities suggested by the triangle structure, we would do well to focus on what might be called *ten-structured thinking*, a focus on the patterns in our number system and how numbers are related to the next or previous multiples of 10, 25, and 100. Although this is my definition, phrase "ten-structured thinking" is borrowed from Karen Fuson's project at Northwestern University as described in *Making Sense* (Hiebert, et al, 1997). The basic ideas are borrowed from the authors of *Developing Mathematical Ideas* (Schifter, Bastable, & Russell, 1999) in the *Building a System of Tens* module. Ideas are also borrowed from the

Standards-based curriculum, *Investigations in Number, Data, and Space.* With ten-structured thinking, 37 is thought of as being 3 away from 40, 13 from 50 and 63 from 100. Rather than focus on the structure of 37 as 3 tens and 7 ones, students might be helped to think about it as 30 and 7 (this is a real difference) and also 25 and 12 more. Similarly, 413 is 400 and 13 more and 592 is 8 away from 600. What might be important about 276 is that it is one more than 275 because multiples of 25 are very easy to work with. We already know a great deal about this approach to number. For example, in the development of addition facts, we encourage a strategy of using ten for facts that have an 8 or 9 or possibly even a 7. The ten-frame has proven to be extremely valuable in this development. Even here we encourage students to be flexible and use strategies that make sense to them. Some may convert 8 + 6 to 10 + 4. Some may add 10 and 6 and take back 2. A strategy not nearly as popular in the west as it might be is to think of numbers as 5 and some more. Thus, 8 + 6 is two 5s and then 3 + 1.

For subtraction, 10 is used in a similar manner. 14 – 8 might be thought of as adding up through 10: 8 and 2 more is 10 and 4 more is 6. For 15 – 6 one might back down to 10 (take away 5) and then 1 more is 9. In either case, working up or back through ten with basic facts is the same approach that can be used with multidigit numbers.

The power of thinking about small numbers in terms of parts and wholes is well known. Thinking about 8 as 6 and 2, 4 and 4, and so on can easily be extended to multiples of ten. The most powerful part-whole activity is to think about missing parts: 6 and what makes 8. Again, we can easily extend this part-whole thought process to two-digit numbers.

30 and what makes 80?

37 and what makes 100?

46 and what makes 83

Return to the last example and consider the variety of methods that students might easily construct to think about this solution. Whatever strategy is settled on by the individual, it is quite likely that it can be extended to three digits with or without benefit of other number constructs.

How many ways can you think of to solve this problem:

Some may start by adding 3 up to 250 or 53 up to 300. Others will add 300 onto 247 and work in a similar manner from there. Still others will notice that 247 is close to 250 and double 250 and add 100 more. It is not important which method is used. Nor is it important that these things be done mentally. Intermediate steps can be written down to aid in short-term memory. What is significant is to note all of the meaningful and useful methods there are to solve 604 – 247 without the difficulties the typical third grader encounters when "borrowing across zero."

The hundreds chart is another common tool in our classrooms but one that has probably not been exploited to its fullest (except in *Investigations* and the *Everyday Math* curricula). The many patterns in the chart are related to the number-oriented structure of our number system. The chart can also be seen as a folded up number line – one that accentuates the distance from any number to the next multiple of 10. A jump down a row is the same as adding 10 and up a row is ten less. Consider how a child might use the hundreds chart to help think about the sum of 38 and 25.

?

?

?

604 in all

247 ?

These and other methods all utilize the concepts of working with tens – ten-structured thinking. Even the child who counts on 38 by ones from 25 is experiencing each multiple of ten at the ends of the rows. In a classroom discussion, even this inefficient strategy can be connected when the child is ready, to a more efficient use of tens. Finally, the hundreds chart can easily be extended to a 1000s chart – a vertical alignment of 10 consecutive hundreds charts allowing early strategies to be expanded to those involving three-digit numbers.

The ideas mentioned here are only offered to illustrate a more number-oriented approach to place value. You might argue that the focus was on addition and subtraction rather than on place value. I believe that a number-oriented approach to place value is one that fully integrates computation with number development. Not only is there no real reason to teach place value in a static manner prior to computation – the current method in nearly all programs – but also, number concepts that are most important can be enhanced by integrating number with computation. The examples used so far can easily be moved into first grade by using smaller numbers, more multiples of ten, and a heavier reliance on the hundreds board and the ten-frame model. By fourth grade and up, the emphasis would continue to be on computation using flexible strategies for all of the operations. Of course some attention to reading, writing, and conceptualizing large numbers, especially with real-world referents and measures (rather than base-ten pieces) remains important.

Encourage the use of "invented strategies."

The place-value development I've just described suggests an integration of student-invented strategies with the number concepts. In addition to this integrated approach, a significant portion of the grade 2 to 6 curriculum must continue to be on computation. But this focus should be on what are unfortunately referred to as "invented strategies." Carpenter and his colleagues have defined an invented strategy as any method other than the traditional algorithm that does not involve the use of physical materials or counting by ones (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). The examples you have seen so far in this paper all fit this definition. Invented strategies are nearly always number oriented rather than digit oriented, left-handed rather than right handed, and flexible rather than rigid. At least in the early stages of development, invented strategies involve informal methods of written support, especially to aid in reasoning and alleviate short-term memory difficulties. However, for many students, these same strategies often become mental methods. There are no mathematical differences between good invented strategies and those used for mental mathematics. To call these strategies "invented" is unfortunate because it suggests that every student must invent them. Evidence suggests that all of the strategies referred to in this paper and many more are in fact, invented by students. That does not mean that every child must invent every strategy. In a classroom community of learners, invented strategies can and should be shared by students and adopted by others when and if they see how the methods make sense. Students should be encouraged to use whatever methods they wish and to use only methods that they understand and can explain. This means students should be encouraged to "borrow" strategies from their peers. It may be more politically useful to refer to these strategies as “flexible, number-oriented” strategies. However, this phrase does not roll off the tongue easily nor is it the label currently used in the literature. There are various methods that can be used to encourage the use of invented strategies. Presenting computations with well-chosen story problems can help to promote certain strategies. For example, a join-change unknown problem such as the following tends to suggest an add-on approach to finding the difference.

Maggie had 68 cents in her bank when Grandpa gave her some more. When she counter her money she had 93 cents. How much did Grandpa give to Maggie?

If the problem were to find out how much money was left after spending 68 of the 93 cents, more students would be apt to try a take-away method which is much more difficult. A good lesson can easily revolve around one or two problems as long as students are still in the development stages of invented strategies. In addition to the answers, students should write down their methods and be prepared to share them with the class. All students should listen to the methods of their classmates. Occasionally, the teacher might pose a similar computation and have the full class try to use a method that has just been shared. Methods that become popular with the class might even be named and put on a poster along with an example. These and similar methods can help those students who are not the inventors of a strategy, adopt methods developed by classmates. An atmosphere of sharing and evaluation by peers allows students to challenge methods that may not be correct or to offer improvements to strategies that work but are not as efficient as they might be. In contrast, when a teacher suggests a strategy, students tend to accept it as the right or best way to proceed – even when the students do not understand the idea. This is the same difficulty that is associated with the traditional algorithms that must come from a book or from the teacher. Every child may not invent invented strategies, but in most instances the class invents them. One of the jobs of a good teacher in the development of invented strategies is to be sure that all students have the opportunity to observe ideas of their classmates in a manner that is easily understood. Often it is useful for the teacher to write on the board the details of the procedure being shared by a student. In contrast to a student writing on the board, when the teacher records the steps being shared they are more likely to be clear and much less time is consumed. The teacher can also use this approach to ask students for clarification or to articulate important steps.

An especially useful recording device for classroom sharing is the open number line that has been described by Fosnot and Dolk (2000) as well as others. This is a simple technique that amounts to illustrating each step on a number line. The only numbers indicated on the line are those articulated by the student. For example, a student might share this approach to 4 x 68:

I used 70s because they were easier than 68s. First I did 70 and 70 is 140. Then I doubled 140 to get 280.

(Teacher – “Why did you double 140?”)

Because that would make four 70s and I already had two 70s. Then I had to take off four sets of 2 because I used 70 instead of only 68. That got me to 272.

It is reasonable to ask if there are instructional methods that will help students develop reasonable strategies for all four operations. Let's take each operation in turn.

70
70
140
70
70
140
280
70
70
140
280
272
double 140
double 140

For addition:

We've already seen several methods for two-digit addition and there are several more. Consider these sums and think of ways they might each be done differently:

38 + 46
28 + 75
486 + 357
99 + 83
37 + 64
286 + 128 + 437

The point to be made is that strategies such as using a close number and then adjusting (99 + 83), using multiples of 25 (28 + 75), and using parts of 100 (37 + 64), are all possibilities that can and should be adjusted to the numbers involved. It is also interesting to consider how easily each of the above computations can be done mentally.

For subtraction:

84 – 6
73 – 46
67 – 20
503 - 267

Again, strategies can and should adapt to the numbers involved. When the subtrahend is a single digit or a multiple of ten or 100, it is generally easy to use a take-away thought process. On the other hand, for the more general two-digit or three-digit subtraction situations, an add-up process is almost always easy to use and significantly easier than take-away and even easier and quicker than the traditional algorithm.

For multiplication:

Students can initially learn to take numbers apart and meaningfully use the distributive property by first exploring products with reasonably easy numbers:

60 x 4

35 x 6
120 x 7
17 x 20
From these easier numbers, students can expand their skills to numbers that are not so "nice."
38 x 7
49 x 6
124 x 7
403 x 5
19 x 30
As with all invented strategies, students can be helped by having problems presented as story problems. Experienced teachers will need to work at moving students from inefficient methods that rely on addition to multiplicative methods. The use of cluster problems as found in the *Investigations* curriculum is one of several proven methods of moving students to more efficient multiplication strategies. In the cluster approach, students are given several simpler, related problems to solve first and are told to use the initial results to help solve the final problem. Students are encouraged to add additional problems to the cluster as they wish. Here are two examples:
10 x 15
9 x 30
9 x 15
10 x 32
20 x 15
20 x 32
29 x 15
19 x 32
Soon students are challenged to make up their own cluster of problems for a given product – that is to invent a useful strategy for multiplication. Finally, the area model for multiplication has long been used in textbooks. However, if students are not told how the area approach is connected to specific steps of an algorithm, their attention is drawn more directly and purposefully to an understanding of the distributive property. For two-digit by two-digit problems, students can be given an empty rectangle of given dimensions and simply asked to determine how many units will fit inside. With the use of standard base-ten pieces, students will soon construct the four usual partial products beginning with the largest section. This process they can easily develop into a strategy that is very close to the standard algorithm.
40 x 30 is 1200
6 x 30 is 180
40 x 7 is 280
6 x 7 is 42
That's 1200 Æ 1380Æ
1580Æ 1660Æ 1702

For division:
When students are challenged to solve multiplication story problems involving division, they typically look for a missing factor. Here is an example:

Mary made $216 by cutting 6 neighbors lawns. If each neighbor was charged the same amount, how much did Mary charge for cutting one lawn?
6 x 30 is 180
6 x 40 is 240 (too much)
So 6 x 35 is 210
6 x 5 is 30
One more 6 would make 216.
Mary charged $36 per lawn.
The cluster problem approach is also effective for division. Another method borrowed from *Investigations* offers students a division problem and three different first steps that might be taken. For example, to solve 2417 ÷ 8, you might begin with one of these:
100 x 8
2400 ÷ 8
300 x 8
Each beginning likely evolves into a different sequence of steps or reasoning. Students explore and discuss strategies that begin with different places. Over time, they develop their own flexibility taking numbers apart and using the parts in different ways.
46
37
I am quick to acknowledge that as computations become increasingly tedious, invented strategies become less efficient. Advocates of invented methods are not searching for strategies that are applicable to all computations. Rather, the goal is flexible, efficient methods for reasonable computations. There is no rational reason for including products such as 473 x 68 and quotients such as 1482 ÷ 726 in a modern curriculum.
ADVANTAGES: THERE ARE SIGNIFICANT ADVANTAGES TO FLEXIBLE, NUMBER-ORIENTED "INVENTED STRATEGIES."
Not only is it quite reasonable to believe that students can and will develop adequate computation techniques for all four operations, there are benefits from this approach that simply do not accrue with the traditional algorithms.

Students make fewer errors .
Research indicates that students using methods they understand make many fewer errors than when strategies are learned without understanding. Earlier we noted that even after decades of good intentions with the standard algorithms, far too many students do not understand the concepts that support them. Not only do these students make errors, but the errors are often systematic and difficult to remediate. Errors with invented strategies are less frequent and almost never systematic.

Less reteaching is required.
In any area of mathematics, reteaching is required when student have forgotten what they have learned earlier or when erroneous procedures must be corrected. The cause of these difficulties can be almost completely traced to a lack of a rich understanding. When procedures are connected to a rich web of ideas – well understood – they tend to be remembered longer and are easily retrieved when temporarily lost due to misuse. Teachers often complain that students' early efforts with alternative strategies are slow and time consuming. Young children often have difficulty articulating their thought processes, especially when the ideas and processes they are using are in the formative

stages. The time-consuming struggle in these early stages, however, results in ideas that are meaningful, well integrated in a web of ideas that are robust and long lasting. An increase in development time is more than made up for with a significant decrease in the need for reteaching and remediation.

Students develop number sense.
"More than just a means to produce answers, computation is increasingly seen as a window on the deep structure of the number system" (*Adding It Up,* p. 182). Although some have argued for the value of an analysis if the traditional algorithms, student development and use of number-oriented, flexible algorithms offers a students a rich view of the numbers. Analysis of traditional algorithms necessarily focuses on the 10-to-1 relationship of the values of adjacent place values – trading 10 ones for a ten or 1 ten for 10 ones. This understanding of place value is certainly nice mathematics; it does not translate into a feeling for flexible methods of separating and combining numbers that is a hallmark of all flexible, number-oriented strategies.

Invented strategies are the basis for mental computation and estimation.
As noted earlier, the attributes of all invented strategies are exactly those of mental methods. When invented strategies are the norm for computation, there is no need to teach other methods or even to talk about mental computation as if it were a separate skill. Often, students who have been taught to record their thinking with invented strategies or to write down intermediate steps, will ask if this writing is really required since they find they can do the procedures more efficiently when they are done mentally. As a computational alternative to getting exact answers, estimation does involve a separate set of skills. For example, choosing simpler but close numbers to use in computation, recognizing compatible combinations, adjusting estimates to account for the change in numbers used, and recognizing if an estimate is higher or lower than the exact result are all important skills related to estimation. The development of flexible, number-oriented strategies plays a significant role in most of these skills. Understanding how numbers are related to the next or previous multiple of 10, 25, or 100 is a foundation for selecting useful numbers to use in an estimation computation. And all estimation does involve a computation that is usually done mentally. Traditional algorithms have no place in these computations.

Flexible methods are often faster than the traditional algorithms.
Consider the product 64 x 8. A simple invented strategy might involve 60 x 8 = 480 and 8 x 4 = 32. The sum of 480 and 32 is 500 + 12 more – 512. This is easily done mentally, or even with some recording, in much less time than writing the 8 below the 64, recording the 2 from 32 and carrying the 3. Then 8 x 6 is 48 and 3 is 51. To see the answer, the 51 is recorded next to the 2. Even quite literate adults are tied to this pencil-and-paper procedure that rarely gets very fast. One who has become adept with non-standard methods will consistently perform addition and subtraction computations more quickly than those using a traditional algorithm. Invented strategies for addition and subtraction nearly always become mental strategies when used regularly.

Algorithm invention is itself a significantly important process of "doing mathematics."
All students who invent a strategy for computing, or who adopt a strategy from a classmate, are involved intimately in the process of inventing an algorithmic procedure. These students come to realize that procedures for difficult tasks can be devised and many develop a confidence in their own ability to do so. This development of procedures is a process that traditionally has been hidden from elementary school students. By opening up this aspect of mathematics early on, a significantly different and valuable view of "doing mathematics" will be opened to young children.

QUESTIONS AND ISSUES

I am fully aware that the acceptance of the proposition to abandon the traditional computational algorithms will be difficult for many regardless of the arguments that may be presented in favor of it. If this paper can serve to further a discussion of issues in a thoughtful manner, then it will have served its purpose. I will offer my answers to a few of the most common questions I've encountered and will articulate some issues that require further thought, discussion, and research.

Why do students seem to "latch on to" the traditional algorithms?
Even in classrooms where teachers work hard at the development of alternative strategies, the traditional algorithms will appear. Students get them from well-meaning parents, older siblings, and even other teachers. Often these students will, if given the opportunity, utilize their newly found traditional methods rather than continue use or development of alternative methods. It is reasonable to ask, "Why?" One hypothesis is found in the fact that the traditional algorithms, when compared to emerging invented strategies, require much less cognitive effort. Students need only to utilize their knowledge of facts and follow the rules. In contrast, their teacher is suggesting that they attempt to devise their own methods and wrestle with number concepts that also may be only emerging or even lacking. Teacher skill must be brought to the issue of helping students see the many rewards of alternative strategies. For young students these include the thrill of accomplishment, the satisfaction of real understanding, and with practice, speed at getting results.
A second hypothesis is that students see the traditional algorithms as those used by adults and that come from adults or others they respect. This may attach an aura to the traditional methods as the "real" or ultimately correct way to compute. It is difficult to ignore the power of adding "the way my dad taught me." Countering this appeal of revered authority again requires real skill – focusing students on the interesting aspects of the process rather than the more mundane goal of getting answers.

What constitutes an acceptable level of efficiency or computational proficiency?
There is ample evidence that for computations such as 368 + 473 or 724 x 6 alternative strategies will provide at least the same efficiency as standard algorithms for most students. This level of efficiency may not emerge quite as quickly as it does with the traditional strategies. As the level of computational complexity increases, efficiency or speed will certainly decrease. For example, 587 x 36 may be a computation for which invented strategies tend to be slow or tedious – as it is with the traditional algorithm.

A less than precise answer to the question posed is that a *reasonable* level of proficiency for *reasonable* computations should be a goal by sixth grade. This would likely include for some state curricula, a reduction in the level of computational complexity required of students using by-hand methods. As computations get complex, as they certainly will in rich, realistic applications and problems, students should regularly be encouraged to use a calculator just as sensible adults do in the world outside of schools. It should be a goal that our students acquire a practical ability to judge when to compute by hand, when with a calculator, and when an estimate is adequate.

When should we teach the standard algorithms?
My personal answer to this is difficult for many to swallow. By "stop teaching the traditional algorithms" I really mean just that – don't teach them ever. I believe that if all of the time now devoted to the goal of traditional computational facility were suddenly available in the K-6 curriculum, not nearly all of it would be required to achieve an acceptable level of proficiency for all requirements of the adult world. For those who simply cannot quite swallow this hard line, I offer this alternative: teach the standard algorithms in the seventh or eighth grade. With the increased sense of number that would accrue from the development of invented strategies, it is likely that all four algorithms could be explored in a single unit. Such a unit could profitably include a comparative study of algorithms that would certainly have some benefit.

How will decimal computation be taught if students do not know the traditional algorithms?
All traditional algorithms for decimal computation are simple extensions of algorithms for whole numbers. How will decimal computation be done if students do not have the traditional algorithms? For addition and subtraction, there is no real problem. Assuming the earlier arguments concerning complexity, computations such as 43.6 – 27.8 can easily be done with a variety of invented strategies.
For multiplication, I would offer the same approach that I've advocated for the past 15 years or so – perform the computation as if there were no decimals in the factors. Then place the decimal point by estimation. For example, 34 x 72 is 2100 + 280 + 68 or 2448. The products 3.4 x 0.72, 0.34 x 0.72 and 3.4 x 7.2 all have the same digits with a decimal inserted somewhere. This fact alone is mathematically interesting. For what is probably the most challenging of these three products, 0.34 x 0.72, there are several paths to the necessary estimate. For example, since 0.34 is about 1/3, we want about a third of 0.72. The answer must be 0.2448 since any other decimal placement makes no sense. Try placing the decimal in the other products using estimation.
Division can be handled in exactly the same manner - place the decimal through estimation after computing without it. The issue of carrying out division of whole numbers to one or more decimal places must also be discussed. This response to the issue of decimals will not satisfy everyone. Nor does it have to. Wise minds and careful thought given to the issue will produce several additional options. For both decimal *and* whole numbers, the bottom line revolves around the issue of computational complexity required by the curriculum. I am confident that if we carefully examine what adults do with computation outside their work environments that the levels of complexity observed there will easily be accommodated without the traditional algorithms.

CHALLENGES

This paper has focused on why we don't need the traditional algorithms and has explored the benefits of practical strategies that can replace them. A full curricular abandonment of the traditional algorithms faces a number of obstacles not addressed by these arguments and will require significant efforts for any district or state that chooses that path. Some of these challenges are listed here without much elaboration.

• A major change in curricular materials would be required – a change not likely to be taken up by publishing companies without full expectation that such curricula will be adopted. That is, change must come from bold leadership, not textbooks.

• Most adults and therefore teachers do not currently have the skills themselves of alternative, number-oriented strategies. Both in-service teacher education as well as pre-service training would need to include substantial attention to helping these teachers develop these skills as well a methods for teaching them.

• For most of today's parents, mathematics in the elementary school includes without question a substantial development with computational skill. When students are not being taught the traditional algorithms, parents are upset and demand explanations. A full abandonment of the traditional algorithms would absolutely require a significant effort at parent education. Parents have a right to expect that their children are being fully prepared for the real world outside of schools.

• TRADITION! The traditional algorithms are not called "traditional" for nothing. Perhaps no other topic is so engrained in the elementary mathematics curriculum. This tradition influences parents and teachers as well as reformers. Tackling this change is to take on over a century of tradition, a task not easily achieved.

This same list of challenges also articulates the reasons why we persist in teaching algorithms that are out dated, problematic, and not needed. To the challenges of tradition, parent pressures, and teacher training, we can also add the pressures of high-stakes testing and a fear of change. Yes, these are all real obstacles and real challenges. However, it is time to think clearly about the traditional algorithms and to answer honestly the following two questions:

• Do the minimal advantages of the traditional algorithms really outweigh the many negatives?

• Are the traditional algorithms really necessary or are we taken in by our own fears and lack of understanding of the alternatives?

What I ask of anyone who reads this paper or engages in a discussion of the value of the algorithms is that arguments on both sides are considered honestly and openly; that fears and tradition not cloud the issue; and that giving children an increased opportunity to think and to develop true number sense be an overarching consideration.

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